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A conformal field theory description of magnetic flux fractionalization in Josephson junction ladders*

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Abstract. We show how the recently proposed effective theory for a Quantum Hall system at “paired states” filling $\nu = 1$ [1,2], the twisted model (TM), well adapts to describe the phenomenology of Josephson Junction ladders (JJL) in the presence of defects. In particular it is shown how naturally the phenomenon of flux fractionalization takes place in such a description and its relation with the discrete symmetries present in the TM. Furthermore we focus on “closed” geometries, which enable us to analyze the topological properties of the ground state of the system in relation to the presence of half flux quanta.

PACS. 11.25.Hf Conformal field theory, algebraic structures – 02.20.Sv Lie algebras of Lie groups – 03.65.Fd Algebraic methods

1 Introduction

Arrays of weakly coupled Josephson junctions provide an experimental realization of the two dimensional (2D) XY model physics. A Josephson junction ladder (JJL) is the simplest quasi-one dimensional version of an array in a magnetic field [3]; recently such a system has been the subject of many investigations because of its possibility to display different transitions as a function of the magnetic field, temperature, disorder, quantum fluctuations and dissipation. In this paper we focus on the phenomenon of fractionalization of the flux quantum $hc/2e$ in a fully frustrated JJL, the basic question being how the phenomenon of Cooper pair condensation can cope with properties of charge (flux) fractionalization, typical of a low dimensional system with a discrete Z_2 symmetry.

We must recall that charge fractionalization has been successfully hypothesized by R. Laughlin to describe the ground state of a strongly correlated 2D electron system, a quantum Hall fluid, at fractional fillings $\nu = \frac{1}{2p+1}$, $p = 1, 2, \ldots$. In such a system charged excitations are present with fractional charge (anyons) and elementary flux $\frac{hc}{e}$. Furthermore the phenomenon of fractionalization of the elementary flux has been found in the description of a quantum Hall fluid at non standard fillings $\nu = \frac{m}{mp+2}$ [1,2], within the context of 2D Conformal Field Theories (CFT) with a Z_m twist.

In references [4,5] it has been shown that the presence of a Z_2 symmetry accounts for more general boundary conditions for the propagating electron fields which arise in quantum Hall systems in the presence of impurities or defects. Furthermore such a symmetry is present also in the fully frustrated XY (FFXY) model or equivalently, see references [6,7], in two dimensional Josephson junction arrays (JJA) with half flux quantum $\frac{1}{2}\frac{hc}{2e}$ threading each square cell and accounts for the degeneracy of the ground state.

It is interesting to notice that it is possible to generate non trivial topologies, i.e. the torus, in the context of a CFT approach, which allows to construct (see Sect. 4) a ground state wave function, whose center of mass describes a coherent superposition of localized states sharing all the non trivial global properties of the order parameter. In particular for the FFXY model they are shown here to be closely related to the presence of half flux quanta, also viewed as topological defects. Furthermore such a construction allows also to describe the fluctuations in 2D of

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the order parameter and its power law behavior at criticality. That is shown explicitly in Section 3 for the plane geometry, through a Coulomb gas description of logarithmically interacting vortices.

The aim of this paper is to show that the twisted model (TM) well adapts to describe the phenomenology of fully frustrated JJL with a topological defect and to analyze the implications of "closed" geometries on the ground state global properties.

The paper is organized as follows:

- In Section 2 we introduce the physics of a fully frustrated JJL evidencing the underlying Z_2 symmetry and then present the modified ladder with a topological defect.
- In Section 3 we describe the role played by such a symmetry in the construction of the TM model and its relation with the ladder physics. Furthermore the degeneracy of the ground state appears to be closely related to the number of excitations (primary fields) of the CFT description.
- In Section 4 the symmetry properties of the ground state conformal blocks are analyzed and its relation with their topological properties shown.
- In Section 5 a brief summary of the results is presented together with some comments and suggestions.

In the Appendix the TM conformal blocks are explicitly given in terms of its boundary states (BS) content [4,5].

2 Josephson junction ladder with a topological defect

In this section, after describing the general properties of a ladder of Josephson junctions as drawn in Figure 1, we introduce an interaction of the charges (Cooper pairs) with a magnetic impurity (defect), as drawn in Figure 2. With each site i we associate a phase φ_i and a charge $2en_i$, representing a superconducting grain coupled to its neighbors by Josephson couplings; n_i and φ_i are conjugate variables satisfying the usual phase-number commutation relation. The Hamiltonian describing the system is given by the quantum phase model (QPM):

$$H = -\frac{E_C}{2}\sum_i\left(\frac{\partial}{\partial\varphi_i}\right)^2 - \sum_{\langle ij\rangle} E_{ij}\cos\left(\varphi_i - \varphi_j - A_{ij}\right), \quad (2.1)$$

where $E_C = \frac{(2e)^2}{C}$ (C being the capacitance) is the charging energy at site i, while the second term is the Josephson coupling energy between sites i and j and the sum is over nearest neighbors. $A_{ij} = \frac{2\pi}{\Phi_0}\int_i^j A\cdot dl$ is the line integral of the vector potential associated to an external magnetic field B and $\Phi_0 = \frac{hc}{2e}$ is the magnetic flux quantum. The gauge invariant sum around a plaquette is $\sum_p A_{ij} = 2\pi f$

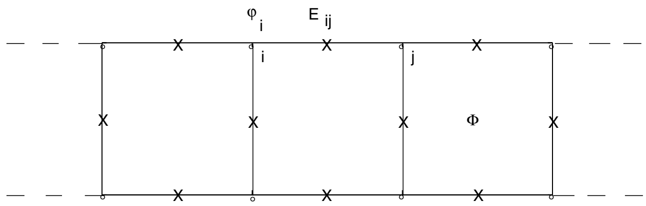

Fig. 1. Josephson junction ladder.

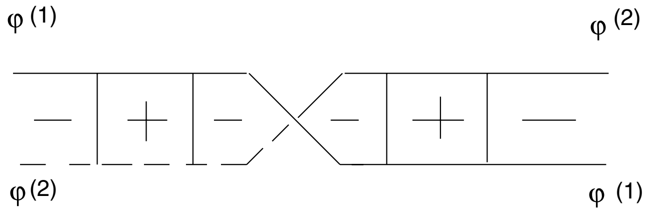

Fig. 2. JJL with an impurity.

with $f = \frac{\Phi}{\Phi_0}$, where Φ is the flux threading each plaquette of the ladder. Let us label the phase fields on the two legs with $\varphi_i^{(a)}$, $a = 1, 2$ and assume $E_{ij} = E_x$ for horizontal links and $E_{ij} = E_y$ for vertical ones. Let us also make the gauge choice $A_{ij} = +\pi f$ for the upper links, $A_{ij} = -\pi f$ for the lower ones and $A_{ij} = 0$ for the vertical ones, which corresponds to a vector potential parallel to the ladder and taking opposite values on upper and lower branches.

Thus the effective quantum Hamiltonian (2.1) can be written as [3]:

$$-H = \frac{E_C}{2}\sum_i\left[\left(\frac{\partial}{\partial\varphi_i^{(1)}}\right)^2 + \left(\frac{\partial}{\partial\varphi_i^{(2)}}\right)^2\right] + \sum_i\left[E_x\sum_{a=1,2}\cos\left(\varphi_{i+1}^{(a)} - \varphi_i^{(a)} + (-1)^a\pi f\right) + E_y\cos\left(\varphi_i^{(1)} - \varphi_i^{(2)}\right)\right]. \quad (2.2)$$

The correspondence between such Hamiltonian and our TM model can be best shown performing the change of variables: $\varphi_i^{(1)} = X_i + \phi_i$, $\varphi_i^{(2)} = X_i - \phi_i$, so equation (2.2) can be cast in the form:

$$-H = \frac{E_C}{2}\sum_i\left[\left(\frac{\partial}{\partial X_i}\right)^2 + \left(\frac{\partial}{\partial\phi_i}\right)^2\right] + \sum_i\left[2E_x\cos\left(X_{i+1} - X_i\right)\cos\left(\phi_{i+1} - \phi_i - \pi f\right) + E_y\cos\left(2\phi_i\right)\right], \quad (2.3)$$

where X_i, ϕ_i (i.e. $\varphi_i^{(1)}$, $\varphi_i^{(2)}$) are only phase deviations of each order parameter from the commensurate phase and should not be identified with the phases of the superconducting grains [3].

When $f = \frac{1}{2}$ and $E_C = 0$ (classical limit) the ground state of the 1D frustrated quantum XY (FQXY) model displays — in addition to the continuous $U(1)$ symmetry of the phase variables — a discrete Z_2 symmetry associated with an antiferromagnetic pattern of plaquette chiralities $\chi_p = \pm 1$, measuring the two opposite directions of the supercurrent circulating in each plaquette. Thus it has two symmetric, energy degenerate, ground states characterized by currents circulating in the opposite directions in alternating plaquettes. For small E_C there is a gap for creation of kinks in the antiferromagnetic pattern of χ_p and the ground state has quasi long range chiral order. The evidence for a chiral phase in Josephson junction ladders has been investigated in reference [8] while a field theoretical description of chiral order is developed in [9].

Performing the continuum limit of the Hamiltonian (2.3):

$$-H = \frac{E_C}{2} \int dx \left[\left(\frac{\partial}{\partial X} \right)^2 + \left(\frac{\partial}{\partial \phi} \right)^2 \right] + \int dx \left[E_x \left(\frac{\partial X}{\partial x} \right)^2 + E_x \left(\frac{\partial \phi}{\partial x} - \frac{\pi}{2} \right)^2 + E_y \cos(2\phi) \right] \tag{2.4}$$

we see that the X and ϕ fields are decoupled. In fact the X term of the above Hamiltonian is that of a free quantum field theory while the ϕ one coincides with the quantum sine-Gordon model. Through an imaginary-time path-integral formulation of such a model [10] it can be shown that the 1D quantum problem maps into a 2D classical statistical mechanics system, the 2D fully frustrated XY model, where the parameter $\alpha = \left(\frac{E_x}{E_C} \right)^{1/2}$ plays the role of an inverse temperature [3]. We work in the regime $E_x \gg E_y$ where the ladder is well described by a CFT with central charge $c = 2$.

We are now ready to introduce the modified ladder as represented in Figure 2. In order to do so let us first require the compactification of the $\varphi^{(a)}$ variables in order to recover the angular nature of the up and down fields. In such a way the XY-vortices, causing the Kosterlitz-Thouless transition, are recovered. Also let us indicate the compactified phases $\varphi^{(1)}, \varphi^{(2)}$ as $\varphi_L^{(1)}, \varphi_R^{(2)}$ respectively (where L, R stay for left, right components). As a second step let us introduce at point $x = 0$ a magnetic impurity which couples the up and down phases through its interaction with the Cooper pairs of the two legs (see Fig. 2). In the limit of strong coupling, that is in the full screening case, such an interaction gives rise to non trivial boundary conditions for the fields [4]:

$$\varphi_L^{(1)}(x = 0) = \mp \varphi_R^{(2)}(x = 0) - \varphi_0. \tag{2.5}$$

It is interesting to notice that such a condition is naturally satisfied by the twisted field $\phi(z)$ of our TM model (see Eq. (3.8)). Furthermore such a field describes both the left moving component $\varphi_L^{(1)}$ and the right moving one $\varphi_R^{(2)}$, which naturally appear in a folded description of a system with a boundary. In fact our TM results in a chiral description of the system just described, in terms of the chiral fields X and ϕ (see Eqs. (3.7, 3.8)). Further details on such an issue are given in Section 3 and in the Appendix, where the relevant chiral fields $\varphi_{e,o}^{(a)}(x) = \varphi_L^{(a)}(x) \pm \varphi_R^{(a)}(-x)$, $a = 1, 2$, which emerge from such conditions, are explicitly constructed, by using the folding procedure [4,5]. In particular we adopt the m-reduction technique [2] which accounts for these non trivial boundary conditions for the JJ ladder due to the presence of a topological defect. Furthermore its realization on closed geometries could be relevant for the description of JJAs with non trivial topologies, which are believed to provide a physical implementation of an ideal quantum computer [11] because of the topological ground state degeneracy which appears to be "protected" from external perturbations [12,13].

In a forthcoming paper [7] we will be also studying in detail two dimensional systems with frustration, the fully frustrated XY model and a two dimensional array of Josephson junctions in an external magnetic field with half flux quantum per cell. Such frustrated systems represent a two dimensional generalization of the linear chain of frustrated plaquettes considered here. Furthermore the phase diagram of such systems [14,15] can be simply understood within our TM description. Recently conformal field theory techniques have been applied as well [6,16,17]; our work follows such a line.

3 The twisted model

We are now ready to show the main steps of our construction.

1. We first construct the bosonic theory, i.e. the TM, and show that its energy momentum tensor fully reproduces the Hamiltonian of equation (2.4) for the JJL. That allows us to describe the JJL excitations in terms of the primary fields $V_\alpha(z)$ given later on in this section and in Section 4 for the torus topology.
2. Then by using standard conformal field theory techniques we show that it is possible to construct the N-vertices correlator for the torus topology in 2D (basically by letting the edge to evolve in "time" and to interact with external vertex operators placed at different points). Throughout this paper we will assume that a suitable correlator is apt to describe the ground state wave function of the JJA at $T = 0$ temperature. We must notice that such an assumption is supported by the plasma description of the system ground state on the plane, given later on in this section. An analysis of the symmetry properties of its center of mass wave function (conformal blocks), which emerge in the presence of vortices carrying half quantum of flux ($\frac{1}{2}\left(\frac{hc}{2e}\right)$), will be given in Section 4.

In this section we recall those aspects of the TM which are relevant for the fully frustrated ($f = \frac{1}{2}$) JJL presented in the previous section. We focus on the m-reduction procedure [2] for the special $m = 2$ case (see Ref. [1] for the general case), since we are interested in a system with a Z_2 symmetry. We showed in references [2,4] that such a theory well adapts to describe a system consisting of two parallel layers of 2D electrons gas in a strong perpendicular magnetic field coupled via a defect line (a topological defect or topological twist). The two layers edges appear coupled at a contact point carrying a magnetic impurity (twist). The bulk electrons isospin interacts with the magnetic impurity and in the limit of strong coupling non trivial boundary conditions, of the Z_2 type in the considered case, for the relevant fields emerge. In this paper we choose the "bosonic" theory, which well adapts to the description of a system with Cooper pairs of electric charge $2e$ in the presence of a topological defect, i.e. a fully frustrated JJL. As pointed out in the previous section, its ground state can be viewed as a sequence of opposite current chiralities in adjacent plaquettes, in close analogy with the checkerboard ground state of the two dimensional JJAs [18]. To each of the two legs (edges) of the ladder we assigned a chirality, making a correspondence between up-down leg and left-right chirality states. In the Appendix each phase field $\varphi^{(a)}$ is written as a sum of two fields of opposite chirality defined on an half-line, because of the presence of a defect at $x = 0$. Within a "bosonization" framework it is shown there how it is possible to reduce to a problem with two chiral fields $\varphi_e^{(a)}$, $a = 1, 2$, each defined on the whole x-axis, and the corresponding dual fields. Now we identify in the continuum such chiral phase fields $\varphi_e^{(a)}$, $a = 1, 2$, each defined on the corresponding leg, with the two chiral fields $Q^{(a)}$, $a = 1, 2$ of our CFT, the TM, with central charge $c = 2$. For clarity sake we must observe that such an identification strongly relies on the fact that the logarithmic phase fluctuations of the order parameter in 2D can be fully expressed in terms of correlators of the CFT $Q^{(a)}(z)$ fields.

In order to construct the fields $Q^{(a)}$ for the TM, we start from a bosonic CFT with $c = 1$ described in terms of a scalar chiral field Q compactified on a circle with radius $R^2 = 2$. It is explicitly given by:

$$Q(z) = q - i\,p \ln z + i \sum_{n\neq 0} \frac{a_n}{n} z^{-n} \tag{3.6}$$

with a_n, q and p satisfying the commutation relations $[a_n, a_{n'}] = n\delta_{n,n'}$ and $[q, p] = i$; its primary fields are the vertex operators $U^{\alpha_l}(z) =: e^{i\alpha_l Q(z)} :$ where $\alpha_l = \frac{l}{\sqrt{2}}$, $l = 1, 2$. It is possible to give a plasma description through the relation $|\psi|^2 = e^{-\beta H_{\mathit{eff}}}$ where $\psi(z_1, ..., z_N) = \langle N\alpha_l| \prod_{i=1}^N U_l^{\alpha}(z_i)|0\rangle = \prod_{i<j=1}^N (z_i - z_j)^{\frac{l^2}{2}}$ is the ground state wave function. It can be immediately seen that $H_{\mathit{eff}} = -l^2 \sum_{i<j=1}^N \ln|z_i - z_j|$ and $\beta = \frac{2}{R^2} = 1$, that is only vorticity $v = 1, 2$ vortices are present in the plasma.

From such a CFT (mother theory), using the m-reduction procedure, which consists in considering the subalgebra generated only by the modes in equation (3.6) which are a multiple of an integer m, we get a $c = m$ orbifold CFT (daughter theory, i.e. the TM). Then the fields in the mother CFT can be organized into components which have well defined transformation properties under the discrete Z_m (twist) group, which is a symmetry of the TM. By using the mapping $z \to z^{1/m}$ and by making the identifications $a_{nm+l} \longrightarrow \sqrt{m}\,a_{n+l/m}$, $q \longrightarrow \frac{1}{\sqrt{m}} q$ the $c = m$ CFT (daughter theory) is obtained.

Its primary fields content, for the special $m = 2$ case, can be expressed in terms of a Z_2-invariant scalar field $X(z)$, given by

$$X(z) = \frac{1}{2}\left(Q^{(1)}(z) + Q^{(2)}(z)\right), \tag{3.7}$$

describing the continuous phase sector of the new theory, and a twisted field

$$\phi(z) = \frac{1}{2}\left(Q^{(1)}(z) - Q^{(2)}(z)\right), \tag{3.8}$$

which satisfies the twisted boundary conditions $\phi(e^{i\pi}z) = -\phi(z)$ [1]. More explicitly such a field can be written in terms of the left and right moving components $\varphi_L^{(1)}$, $\varphi_R^{(2)}$; then the boundary conditions given in equation (2.5) are fully described by the boundary conditions for ϕ. This will be more evident for closed geometries, i.e. for the torus case, where the magnetic impurity gives rise to a line defect so allowing us to resort to the folding procedure and introduce boundary states [4,5] (see Appendix for details).

Furthermore the fields in equations (3.7, 3.8) coincide with the ones introduced in equation (2.4). In fact the energy momentum tensor for such fields given in equation (3.13) fully reproduces the second quantized Hamiltonian of equation (2.4) as we will see at the end of the section. Let us notice that the angular nature of the phase fields in our theory takes into account also the presence of vortices, i.e. topological excitations which cause a Kosterlitz-Thouless transition, which are responsible for the periodicity of the phase diagram and which were not considered in the analysis of reference [3].

The whole TM theory decomposes into a tensor product of two CFTs, a twisted invariant one with $c = \frac{3}{2}$ and the remaining $c = \frac{1}{2}$ one realized by a Majorana fermion in the twisted sector. In the $c = \frac{3}{2}$ sub-theory the primary fields are composite vertex operators $V(z) = U_X^{\alpha_l}(z)\,\psi(z)$ or $V_{qh}(z) = U_X^{\alpha_l}(z)\,\sigma(z)$, where

$$U_X^{\alpha_l}(z) = \frac{1}{\sqrt{z}} : e^{i\alpha_l X(z)} : \tag{3.9}$$

is the vertex of the continuous sector with $\alpha_l = \frac{l}{2}$, $l = 1, ..., 4$ for the $SU(2)$ Cooper pairing symmetry used here. In order to give a physical meaning to the vertices shown in equation (3.9) we consider at the end of this Section the N-point correlator on the plane. As a result we obtain a plasma description with logarithmically interacting vortices in the ground state of the JJ system, so reproducing the properties of the fluctuations of the order parameter. For clarity sake we must further comment that, even though the global properties of the order parameter are already encrypted in the non trivial braiding relations of the above vertices (through the Bohm-Aharonov phase picked up when exchanging them), it is more instructive for the condensed matter physicist to analyze them for the non trivial torus topology. In fact by construction the non perturbative ground state emerges there naturally as a coherent superposition of localized states, together with its global properties and it will be presented in Section 4.

The corresponding energy-momentum tensor is:

$$T_X(z) = -\frac{1}{2}(\partial X)^2. \tag{3.10}$$

Regarding the other component, the highest weight state in the isospin sector, it can be classified by the two chiral operators:

$$\psi(z) = \frac{1}{2\sqrt{z}}\left(:e^{i\sqrt{2}\phi(z)}: + :e^{i\sqrt{2}\phi(-z)}:\right),$$

$$\overline{\psi}(z) = \frac{1}{2\sqrt{z}}\left(:e^{i\sqrt{2}\phi(z)}: - :e^{i\sqrt{2}\phi(-z)}:\right); \tag{3.11}$$

which correspond to two $c = \frac{1}{2}$ Majorana fermions with Ramond (invariant under the Z_2 twist) or Neveu-Schwartz (Z_2 twisted) boundary conditions [1,2] in a fermionized version of the theory. Let us point out that the energy-momentum tensor of the Ramond part of the isospin sector develops a cosine term:

$$T_\psi(z) = -\frac{1}{4}(\partial\phi)^2 - \frac{1}{16z^2}\cos\left(2\sqrt{2}\phi\right). \tag{3.12}$$

The Ramond fields are the degrees of freedom which survive after the tunnelling and the parity symmetry, which exchanges the two Ising fermions, is broken.

So the whole energy-momentum tensor within the $c = \frac{3}{2}$ sub-theory is:

$$T = T_X(z) + T_\psi(z) = -\frac{1}{2}(\partial X)^2 - \frac{1}{4}(\partial\phi)^2 - \frac{1}{16z^2}\cos\left(2\sqrt{2}\phi\right). \tag{3.13}$$

The correspondence with the Hamiltonian of equation (2.4) is more evident once we observe that the isospin current $\partial\phi$ appearing above coincides with the term $(\partial\phi - \frac{\pi}{2})$ of equation (2.4), since the $\frac{\pi}{2}$-term coming from the frustration condition, here it appears as a zero mode, i.e. a classical mode. That is the frustration $\frac{\pi}{2}$ (in general πf) of the ladder cells here in the TM construction is related to the order of the twist Z_2 ($Z_{1/f}$ in the general case). Besides the fields appearing in equation (3.11), there are the $\sigma(z)$ fields, also called the twist fields, which appear in the quasi-hole primary fields $V_{qh}(z)$. Their presence is a peculiarity of the fully frustrated XY model in which they appear at the corner where two domain walls meet [6]. The twist fields have non local properties and decide also for the non trivial properties of the vacuum state, which in fact can be twisted or not in our formalism. Such a property for the vacuum is more evident for the torus topology, where the σ-field is described by the conformal block $\chi_{\frac{1}{16}}$ (see Appendix).

The evidence of a phase transition in ladder systems at $c = \frac{3}{2}$ has been investigated in [19] within a CFT framework. Within this framework the ground state wave function for the plane is described as a correlator of N_{2e} Cooper pairs:

$$\langle N_{2e}\alpha|\prod_{i=1}^{N_{2e}} V(z_i)|0\rangle = \prod_{i<i'=1}^{N_{2e}}(z_i - z_{i'})\,Pf\left(\frac{1}{z_i - z_{i'}}\right) \tag{3.14}$$

where $Pf\left(\frac{1}{z_i - z_{i'}}\right) = \mathcal{A}\left(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}...\right)$ is the antisymmetrized product over pairs of Cooper pairs, so reproducing well known results [20]. In a similar way we also are able to evaluate correlators of N_{2e} Cooper pairs in the presence of (quasi-hole) excitations [1,20] with non Abelian statistics [21].

It is now interesting to notice that the charged contribution appearing in the correlator of N_e electrons is just: $\langle N_e\alpha|\prod_{i=1}^{N_e} U_X^{1/2}(z_i)|0\rangle = \prod_{i<i'=1}^{N_e}(z_i - z_{i'})^{1/4}$, giving rise to a vortices plasma with $H_{eff} = -\frac{1}{4}\sum_{i<j=1}^{N}\ln|z_i - z_j|$ at the corresponding temperature $\beta = \frac{2}{R_X^2} = 2$, that is it describes vortices with vorticity $v = \frac{1}{2}$! From the above relations it clearly emerges how the order parameter fluctuations in 2D are so faithfully reproduced within our TM.

4 Symmetry properties of the TM conformal blocks

In Section 3 we identified our chiral fields $Q^{(a)}$ with the continuum limit of the Josephson phase $\varphi^{(a)}$ defined on the two legs of the ladder respectively and considered non trivial boundary conditions at its ends, so constructing a version in the continuum of the discrete system. Starting from the primary fields $V_\alpha(z)$ given in the previous section we can now construct the non perturbative ground state wave function of the JJ system for the torus topology. It turns out that by construction it results as a coherent superposition of Gaussian states with all the non trivial global properties of the order parameter. In fact by using standard conformal field theory techniques it is now possible to generate the torus topology, starting from the edge theory, just defined in the previous section. That is realized by evaluating the N-vertices correlator

$$\langle n|V_\alpha(z_1)\ldots V_\alpha(z_N)\,e^{2\pi i\tau L_0}|n\rangle, \tag{4.15}$$

where $V_\alpha(z_i)$ is the generic primary field of Section 3 representing the excitation at z_i, L_0 is the Virasoro generator for dilatations and τ the proper time. The neutrality condition $\sum\alpha = 0$ must be satisfied and the sum over the complete set of states $|n\rangle$ is indicating that a trace must be taken. Even though for the present paper it is not necessary to go through such a calculation, it is very illuminating for the non expert reader to pictorially represent the above operation in terms of an edge state (that is a primary state defined at a given τ) which propagates interacting with external fields at $z_1 \ldots z_N$ and finally getting back to itself. In such a way a 2D surface is generated with the torus topology. It is interesting to observe that such a procedure is equivalent to the coherent insertion of correlated relevant vortices (as provided by the CFT description) at positions $z_1 \ldots z_N$, as they appear in the non perturbative ground state of the physical JJ system. From such a picture it is evident then how the degeneracy of the non perturbative ground state is closely related to the number of primary states. Furthermore, since in this paper we are interested in the understanding of the topological properties of the system, we can consider only the center of mass contribution in the above correlator, so neglecting its short distances properties. To such an extent the one-point functions are extensively reported in the following.

On the torus [2] the TM primary fields are described in terms of the conformal blocks of the Z_2-invariant $c = \frac{3}{2}$ subtheory and of the non invariant $c = \frac{1}{2}$ Ising model, so reflecting the decomposition on the plane outlined in the previous section. The following characters

$$\bar{\chi}_0(0|\tau) = \frac{1}{2}\left(\sqrt{\frac{\theta_3(0|\tau)}{\eta(\tau)}} + \sqrt{\frac{\theta_4(0|\tau)}{\eta(\tau)}}\right),$$

$$\bar{\chi}_{\frac{1}{2}}(0|\tau) = \frac{1}{2}\left(\sqrt{\frac{\theta_3(0|\tau)}{\eta(\tau)}} - \sqrt{\frac{\theta_4(0|\tau)}{\eta(\tau)}}\right),$$

$$\bar{\chi}_{\frac{1}{16}}(0|\tau) = \sqrt{\frac{\theta_2(0|\tau)}{2\eta(\tau)}}$$

express the primary fields content of the Ising model with Neveu–Schwartz (Z_2 twisted) boundary conditions, while

$$\chi^{c=3/2}_{(0)}(0|w_c|\tau) = \chi_0(0|\tau)K_0(w_c|\tau) + \chi_{\frac{1}{2}}(0|\tau)K_2(w_c|\tau), \tag{4.16}$$

$$\chi^{c=3/2}_{(1)}(0|w_c|\tau) = \chi_{\frac{1}{16}}(0|\tau)\left(K_1(w_c|\tau) + K_3(w_c|\tau)\right), \tag{4.17}$$

$$\chi^{c=3/2}_{(2)}(0|w_c|\tau) = \chi_{\frac{1}{2}}(0|\tau)K_0(w_c|\tau) + \chi_0(0|\tau)K_2(w_c|\tau) \tag{4.18}$$

represent those of the Z_2-invariant $c = \frac{3}{2}$ CFT. They are given in terms of a "charged" $K_\alpha(w_c|\tau)$ contribution, (see definition given below) and a "isospin" one $\chi_\beta(0|\tau)$, (the conformal blocks of the Ising Model), where $w_c = \frac{1}{2\pi i}\ln z_c$ is the torus variable of "charged" component. Notice that the corresponding argument of the isospin block is $w_n = 0$ everywhere.

In order to understand the physical significance of the $c = 2$ conformal blocks in terms of the charged low energy excitations of the system, let us evidence their electric charge (magnetic flux contents in the dual theory, which is obtained by exchanging the compactification radius $R_e^2 \rightarrow R_m^2$ in the charged sector of the CFT). In order to do so let us consider the "charged" sector conformal blocks appearing in equations (4.16–4.18):

$$K_{2l+i}(w_c|\tau) = \frac{1}{\eta(\tau)}\,\Theta\begin{bmatrix}\frac{2l+i}{4}\\ 0\end{bmatrix}(2w_c|4\tau), \forall\,(l,i) \in (0,1)^2. \tag{4.19}$$

They correspond to primary fields with conformal dimensions

$$h_{2l+i} = \frac{1}{2}\,\alpha^2_{(l,i)} = \frac{1}{2}\left(\frac{2l+i}{2} + 2\delta_{(l+i),0}\right)^2$$

and electric charges $2e(\frac{\alpha_{(l,i)}}{R_X})$, magnetic charges in the dual theory $\frac{hc}{2e}\left(\alpha_{(l,i)}R_X\right)$, $R_X = 1$ being the compactification radius. More explicitly the electric charges (magnetic charges in the dual theory) are the following:

$$\begin{aligned}
&l = 0, \quad i = 0, \quad q_e = 4e, \quad \left(q_m = 2\frac{hc}{2e}\right),\\
&l = 1, \quad i = 0, \quad q_e = 2e, \quad \left(q_m = \frac{hc}{2e}\right),\\
&l = 0, \quad i = 1, \quad q_e = e, \quad \left(q_m = \frac{1}{2}\frac{hc}{2e}\right),\\
&l = 1, \quad i = 1, \quad q_e = 3e, \quad \left(q_m = \frac{3}{2}\frac{hc}{2e}\right).
\end{aligned} \tag{4.20}$$

If we now turn to the whole $c = 2$ theory, the characters of the twisted sector are given by:

$$\begin{aligned}
\chi^+_{(0)}(0|w_c|\tau) &= \bar{\chi}_{\frac{1}{16}}(0|\tau)\left(\chi^{c=3/2}_0(0|w_c|\tau) + \chi^{c=3/2}_2(0|w_c|\tau)\right)\\
&= \bar{\chi}_{\frac{1}{16}}\left(\chi_0 + \chi_{\frac{1}{2}}\right)(K_0 + K_2),
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
\chi^+_{(1)}(0|w_c|\tau) &= \left(\bar{\chi}_0(0|\tau) + \bar{\chi}_{\frac{1}{2}}(0|\tau)\right)\chi^{c=3/2}_1(0|w_c|\tau)\\
&= \chi_{\frac{1}{16}}\left(\bar{\chi}_0 + \bar{\chi}_{\frac{1}{2}}\right)(K_1 + K_3),
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
\chi^-_{(0)}(0|w_c|\tau) &= \bar{\chi}_{\frac{1}{16}}(0|\tau)\left(\chi^{c=3/2}_0(0|w_c|\tau) - \chi^{c=3/2}_2(0|w_c|\tau)\right)\\
&= \bar{\chi}_{\frac{1}{16}}\left(\chi_0 - \chi_{\frac{1}{2}}\right)(K_0 - K_2),
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
\chi^-_{(1)}(0|w_c|\tau) &= \left(\bar{\chi}_0(0|\tau) - \bar{\chi}_{\frac{1}{2}}(0|\tau)\right)\chi^{c=3/2}_1(0|w_c|\tau)\\
&= \chi_{\frac{1}{16}}\left(\bar{\chi}_0 - \bar{\chi}_{\frac{1}{2}}\right)(K_1 + K_3).
\end{aligned} \tag{4.24}$$

Furthermore the characters of the untwisted sector are [2]:

$$\begin{aligned}\tilde{\chi}^{+}_{(0)}(0|w_c|\tau) &= \bar{\chi}_0(0|\tau)\chi^{c=3/2}_{(0)}(0|w_c|\tau)\\ &\quad + \bar{\chi}_{\frac{1}{2}}(0|\tau)\chi^{c=3/2}_{(2)}(0|w_c|\tau)\\ &= \left(\bar{\chi}_0\chi_0 + \bar{\chi}_{\frac{1}{2}}\chi_{\frac{1}{2}}\right)K_0 + \left(\bar{\chi}_0\chi_{\frac{1}{2}} + \bar{\chi}_{\frac{1}{2}}\chi_0\right)K_2,\end{aligned} \tag{4.25}$$

$$\begin{aligned}\tilde{\chi}^{+}_{(1)}(0|w_c|\tau) &= \bar{\chi}_0(0|\tau)\chi^{c=3/2}_{(2)}(0|w_c|\tau)\\ &\quad + \bar{\chi}_{\frac{1}{2}}(0|\tau)\chi^{c=3/2}_{(0)}(0|w_c|\tau)\\ &= \left(\bar{\chi}_0\chi_{\frac{1}{2}} + \bar{\chi}_{\frac{1}{2}}\chi_0\right)K_0 + \left(\bar{\chi}_0\chi_0 + \bar{\chi}_{\frac{1}{2}}\chi_{\frac{1}{2}}\right)K_2,\end{aligned} \tag{4.26}$$

$$\begin{aligned}\tilde{\chi}^{-}_{(0)}(0|w_c|\tau) &= \bar{\chi}_0(0|\tau)\chi^{c=3/2}_{(0)}(0|w_c|\tau)\\ &\quad - \bar{\chi}_{\frac{1}{2}}(0|\tau)\chi^{c=3/2}_{(2)}(0|w_c|\tau)\\ &= \left(\bar{\chi}_0\chi_0 - \bar{\chi}_{\frac{1}{2}}\chi_{\frac{1}{2}}\right)K_0 + \left(\bar{\chi}_0\chi_{\frac{1}{2}} - \bar{\chi}_{\frac{1}{2}}\chi_0\right)K_2,\end{aligned} \tag{4.27}$$

$$\begin{aligned}\tilde{\chi}^{-}_{(1)}(0|w_c|\tau) &= \bar{\chi}_0(0|\tau)\chi^{c=3/2}_{(2)}(0|w_c|\tau)\\ &\quad - \bar{\chi}_{\frac{1}{2}}(0|\tau)\chi^{c=3/2}_{(0)}(0|w_c|\tau)\\ &= \left(\bar{\chi}_0\chi_{\frac{1}{2}} - \bar{\chi}_{\frac{1}{2}}\chi_0\right)K_0 + \left(\bar{\chi}_0\chi_0 - \bar{\chi}_{\frac{1}{2}}\chi_{\frac{1}{2}}\right)K_2,\end{aligned} \tag{4.28}$$

$$\begin{aligned}\tilde{\chi}_{(0)}(0|w_c|\tau) &= \bar{\chi}_{\frac{1}{16}}(0|\tau)\chi^{c=3/2}_{(1)}(0|w_c|\tau)\\ &= \bar{\chi}_{\frac{1}{16}}\chi_{\frac{1}{16}}\left(K_1 + K_3\right).\end{aligned} \tag{4.29}$$

Such a factorization is a consequence of the parity selection rule (m-ality), which gives a gluing condition for the "charged" and "isospin" excitations. The conformal blocks given above represent the collective states of highly correlated vortices, which appear to be incompressible. In order to show the corresponding symmetry properties it is useful to give a pictorial description of the conformal blocks appearing in equation (4.19). To such an extent let us imagine to cut the torus along the A-cycle. The different primary fields then can be seen as excitations which propagate along the B-cycle and interact with the external Cooper pair at point w_c. We can now test the symmetry properties of the characters of the theory (given above) by simply evaluating the Bohm–Aharonov phase they pick up while a Cooper pair is taken along the closed A-cycle. In order to do that, it is important to notice that the transport of the "Cooper pair" from the upper (with isospin up) leg to the down (with isospin down) leg can be realized by a translation of the variables w_c and w_n, which must be identical for the "charged" and the "isospin" sectors. In fact it turns out that the translation with $\Delta w_c = \Delta w_n$ allows us to describe, for example in the twisted sector, the charge transport from leg 1 (isospin up) to leg 2 (isospin down) through the crossing point shown in Figure 2.

So under a 2π-rotation the torus variables transform as $\Delta w_c = \Delta w_n = 1$ and it is easy to check that:

$$\begin{aligned}K_{0,2}(w_c+1|\tau) &= K_{0,2}(w_c|\tau),\\ K_{1,3}(w_c+1|\tau) &= -K_{1,3}(w_c|\tau).\end{aligned} \tag{4.30}$$

Let us observe that the change in sign in the last relation of equation (4.30) is strictly related to the presence in the spectrum of excitations carrying fractionalized charge quanta. Now, turning on also the isospin sector contribution in the Cooper pair transport along the A-cycle, we obtain in a straightforward way:

$$\chi_{0,\frac{1}{2}}(1|\tau) = \chi_{0,\frac{1}{2}}(0|\tau),\ \chi_{\frac{1}{16}}(1|\tau) = i\chi_{\frac{1}{16}}(0|\tau) \tag{4.31}$$

and the same is true for the characters $\bar{\chi}_\beta$. Notice that the phase factor $i = e^{i\pi/2}$ appearing above in the transport of the isospin "cloud" by the $\chi_{\frac{1}{16}}$ character is again due to the presence of a half-flux.

As a result the ground state described by equation (4.29):

$$\tilde{\chi}_{(0)}(0|w_c|\tau) = \bar{\chi}_{\frac{1}{16}}\chi_{\frac{1}{16}}\left(K_1 + K_3\right) \tag{4.32}$$

does not change sign under the transport of a Cooper pair along the closed A-cycle by the amount $\Delta w_c = \Delta w_n = 1$. In fact the negative sign coming from the continuous phase sector is compensated by the negative sign coming from the other sector! Of course the same is true for all the other characters of the untwisted sector, i.e. we cannot trap a half flux quantum in the hole in the untwisted sector.

Instead in the twisted sector the ground state wavefunctions show a non trivial behavior. In fact under $\Delta w_c = \Delta w_n = 1$

$$\begin{aligned}\chi^{\pm}_{(0)}(1|w_c+1|\tau) &= +i\chi^{\pm}_{(0)}(0|w_c|\tau),\\ \chi^{\pm}_{(1)}(1|w_c+1|\tau) &= -i\chi^{\pm}_{(1)}(0|w_c|\tau).\end{aligned}$$

The change in phase given above evidences the presence of a half flux quantum in the hole as it will be clear below. In fact in the twisted case geometry (see Fig. 2) the Cooper pair flows along the ladder and changes isospin in a 2π-period, so implying that in such a case the transport of a Cooper pair from a given point w on the A-cycle to the same point has a 4π-period, that is it corresponds to $\Delta w_c = \Delta w_n = 2$. Under this transformation the characters given above get the following non trivial Bohm-Aharonov phase:

$$\chi^{\pm}_{(0,1)}(2|w_c+2|\tau) = -\chi^{\pm}_{(0,1)}(0|w_c|\tau), \tag{4.33}$$

so explicitly evidencing the trapping of $\frac{1}{2}\left(\frac{hc}{2e}\right)$ in the hole.

It is worthwhile to notice that the properties just discussed are independent of the short distance properties of the vortices plasma, the only crucial requirement for its stability being the neutrality condition.

5 Brief summary with comments

In this paper we presented a simple collective description of a ladder of Josephson junctions with a macroscopic half flux quantum trapped in the hole. It was shown how the phenomenon of flux fractionalization takes place within the context of a 2D conformal field theory with a Z_2 twist, the TM. The presence of a Z_2 symmetry indeed accounts for more general boundary conditions for the fields describing the Cooper pairs propagating on the ladder legs, which arise from the presence of a magnetic impurity strongly coupled with the Josephson phases. For closed geometries and in the limit of the continuum the phase fields $\varphi^{(a)}$ defined on the two legs were identified with the two chiral Fubini fields $Q^{(a)}$ of our TM, and a correspondence between the energy momentum density tensor for such fields (or better the X and ϕ fields of Eqs. (3.7–3.8)) and the Hamiltonian of equation (2.4) traced. For such geometries it was also indicated that the Kosterlitz-Thouless vortices were recovered.

Furthermore it was shown that for closed geometries the JJL with an impurity gives rise to a line defect, which can be turned into a boundary state after employing the folding procedure. That enabled us to derive the low energy charged excitations of the system as provided by our description, with the superconducting phase characterized by condensation of $4e$ charges and gapped $2e$ excitations. Finally, by simply evaluating a Bohm-Aharonov phase, it has been evidenced that non trivial symmetry properties for the conformal blocks emerge due to the presence in the spectrum of fractionalized flux quanta $\frac{1}{2}\left(\frac{hc}{2e}\right)$. As it has been explained before, that signals the presence of a topological defect in the twisted sector of the TM. The question of an emerging topological order in the ground state together with the possibility of providing protected states for the implementation of a solid state qubit has been addressed elsewhere [7,13]. Notice also that the different behavior of the $2e$ and $4e$ excitations is well evidenced by the Bohm-Aharonov phase. Indeed while the transport of a $2e$ along the cycle induces a-1 phase factor, in the $4e$ excitation transport the phase factor is trivial [11]. This is the consequence of the symmetry of the $4e$ with respect to the leg index. Also Josephson junctions ladders with annular geometry have been fabricated within the trilayer Nb/Al-AlO_x/Nb technology and experimentally investigated [22]. So in principle it could be simple to conceive an experimental setup in order to test our predictions.

It is interesting to notice that the presence of a topological defect has been experimentally evidenced very recently for a two layers quantum Hall system, by measuring the conduction properties between two edge states of the system [23].

We conclude by observing that it would be useful to extend our approach to a generic frustration $f=\frac{1}{m}$.

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Appendix: TM boundary states

Let us now recall briefly the TM boundary states (BS) recently constructed in [4].

For closed geometries, that is for the torus, the JJL with an impurity gives rise to a line defect in the bulk. In order to describe it we resort to the folding procedure. Such a procedure is used in the literature to map a problem with a defect line (as a bulk property) into a boundary one, where the defect line appears as a boundary state of a theory which is not anymore chiral and its fields are defined in a reduced region which is one half of the original one. Our approach, the TM, is a chiral description of that, where the chiral ϕ field defined in $(-L/2,\ L/2)$ describes both the left moving component and the right moving one defined in $(-L/2,\ 0)$, $(0,\ L/2)$ respectively, in the folded description [4,5]. Furthermore to make a connection with the TM we consider more general gluing conditions:

$$\phi_L(x=0)=\mp\phi_R(x=0)-\varphi_0$$

the $-(+)$ sign staying for the twisted (untwisted) sector. We are then allowed to use the boundary states given in [24] for the $c=1$ orbifold at the Ising^2 radius. The X field, which is even under the folding procedure, does not suffer any change in boundary conditions [4]. Let us now write each phase field as the sum $\varphi^{(a)}(x)=\varphi_L^{(a)}(x)+\varphi_R^{(a)}(x)$ of left and right moving fields defined on the half-line because of the defect located in $x=0$. Then let us define for each leg the two chiral fields $\varphi_{e,o}^{(a)}(x)=\varphi_L^{(a)}(x)\pm\varphi_R^{(a)}(-x)$, each defined on the whole x-axis [25]. In such a framework the dual fields $\varphi_o^{(a)}(x)$ are fully decoupled because the corresponding boundary interaction term in the Hamiltonian does not involve them [26]; they are involved in the definition of the conjugate momenta $\Pi_{(a)}=\left(\partial_x\varphi_o^{(a)}\right)=\left(\frac{\partial}{\partial\varphi_e^{(a)}}\right)$ present in the quantum Hamiltonian. Performing the change of variables $\varphi_e^{(1)}=X+\phi$, $\varphi_e^{(2)}=X-\phi$ ($\varphi_o^{(1)}=\overline{X}+\overline{\phi}$, $\varphi_o^{(2)}=\overline{X}-\overline{\phi}$ for the dual ones) we get the quantum Hamiltonian (2.4) but, now, all the fields are chiral ones.

It is interesting to notice that the condition (2.5) is naturally satisfied by the twisted field $\phi(z)$ of our twisted model (TM) (see Eq. (3.8)).

The most convenient representation of such BS is the one in which they appear as a product of Ising and $c=\frac{3}{2}$ BS. These last ones are given in terms of the BS $|\alpha\rangle$ for the charged boson and the Ising ones $|f\rangle$, $|\uparrow\rangle$, $|\downarrow\rangle$, according to (see Ref. [27] for details):

$$|\chi_{(0)}^{c=3/2}\rangle=|0\rangle\otimes|\uparrow\rangle+|2\rangle\otimes|\downarrow\rangle \tag{6.34}$$

$$|\chi_{(1)}^{c=3/2}\rangle=\frac{1}{2^{1/4}}\left(|1\rangle+|3\rangle\right)\otimes|f\rangle \tag{6.35}$$

$$|\chi_{(2)}^{c=3/2}\rangle=|0\rangle\otimes|\downarrow\rangle+|2\rangle\otimes|\uparrow\rangle. \tag{6.36}$$

Such a factorization naturally arises already for the TM characters [2].

The vacuum state for the TM model corresponds to the $\tilde{\chi}_{(0)}$ character which is the product of the vacuum state for the $c=\frac{3}{2}$ sub-theory and that of the Ising one. From equations (4.25, 4.27) we can see that the lowest energy state appears in two characters, so a linear combination of them must be taken in order to define a unique vacuum state. The correct BS in the untwisted sector are:

$$|\tilde{\chi}_{((0,0),0)}\rangle = \frac{1}{\sqrt{2}}\left(|\tilde{\chi}^{+}_{(0)}\rangle + |\tilde{\chi}^{-}_{(0)}\rangle\right) = \sqrt{2}(|0\rangle \otimes |\uparrow \bar{\uparrow}\rangle + |2\rangle \otimes |\downarrow \bar{\uparrow}\rangle) \quad (6.37)$$

$$|\tilde{\chi}_{((0,0),1)}\rangle = \frac{1}{\sqrt{2}}\left(|\tilde{\chi}^{+}_{(0)}\rangle - |\tilde{\chi}^{-}_{(0)}\rangle\right) = \sqrt{2}(|0\rangle \otimes |\downarrow \bar{\downarrow}\rangle + |2\rangle \otimes |\uparrow \bar{\downarrow}\rangle) \quad (6.38)$$

$$|\tilde{\chi}_{((1,0),0)}\rangle = \frac{1}{\sqrt{2}}\left(|\tilde{\chi}^{+}_{(1)}\rangle + |\tilde{\chi}^{-}_{(1)}\rangle\right) = \sqrt{2}(|0\rangle \otimes |\downarrow \bar{\uparrow}\rangle + |2\rangle \otimes |\uparrow \bar{\uparrow}\rangle) \quad (6.39)$$

$$|\tilde{\chi}_{((1,0),1)}\rangle = \frac{1}{\sqrt{2}}\left(|\tilde{\chi}^{+}_{(1)}\rangle - |\tilde{\chi}^{-}_{(1)}\rangle\right) = \sqrt{2}(|0\rangle \otimes |\uparrow \bar{\downarrow}\rangle + |2\rangle \otimes |\downarrow \bar{\downarrow}\rangle) \quad (6.40)$$

$$|\tilde{\chi}_{(0)}(\varphi_0)\rangle = \frac{1}{2^{1/4}}\left(|1\rangle + |3\rangle\right) \otimes |D_O(\varphi_0)\rangle \quad (6.41)$$

where we also added the states $|\tilde{\chi}_{(0)}(\varphi_0)\rangle$ in which $|D_O(\varphi_0)\rangle$ is the continuous orbifold Dirichlet boundary state defined in reference [24]. For the special $\varphi_0 = \pi/2$ value one obtains:

$$|\tilde{\chi}_{(0)}\rangle = \frac{1}{2^{1/4}}\left(|1\rangle + |3\rangle\right) \otimes |ff\rangle. \quad (6.42)$$

For the twisted sector we have:

$$|\chi^{+}_{(0)}\rangle = (|0\rangle + |2\rangle) \otimes (|\uparrow \bar{f}\rangle + |\downarrow \bar{f}\rangle) \quad (6.43)$$

$$|\chi^{+}_{(1)}\rangle = \frac{1}{2^{1/4}}\left(|1\rangle + |3\rangle\right) \otimes (|f\bar{\uparrow}\rangle + |f\bar{\downarrow}\rangle) \quad (6.44)$$

$$|\chi^{-}_{(0)}\rangle = (|0\rangle - |2\rangle) \otimes (|\uparrow \bar{f}\rangle - |\downarrow \bar{f}\rangle) \quad (6.45)$$

$$|\chi^{-}_{(1)}\rangle = \frac{1}{2^{1/4}}\left(|1\rangle + |3\rangle\right) \otimes (|f\bar{\uparrow}\rangle - |f\bar{\downarrow}\rangle). \quad (6.46)$$

Now, by using as reference state $|A\rangle$ the vacuum state given in equation (6.37), we compute the chiral partition functions Z_{AB} where $|B\rangle$ are all the BS just obtained [4]:

$$Z_{\langle\tilde{\chi}_{((0,0),0)}||\tilde{\chi}_{((0,0),0)}\rangle} = \tilde{\chi}_{((0,0),0)} \quad (6.47)$$

$$Z_{\langle\tilde{\chi}_{((0,0),0)}||\tilde{\chi}_{((1,0),0)}\rangle} = \tilde{\chi}_{((1,0),0)} \quad (6.48)$$

$$Z_{\langle\tilde{\chi}_{((0,0),0)}||\tilde{\chi}_{((0,0),1)}\rangle} = \tilde{\chi}_{((0,0),1)} \quad (6.49)$$

$$Z_{\langle\tilde{\chi}_{((0,0),0)}||\tilde{\chi}_{((1,0),1)}\rangle} = \tilde{\chi}_{((1,0),1)} \quad (6.50)$$

$$Z_{\langle\tilde{\chi}_{((0,0),0)}||\tilde{\chi}_{(0)}\rangle} = \tilde{\chi}_{(0)} \quad (6.51)$$

$$Z_{\langle\tilde{\chi}_{((0,0),0)}||\chi^{+}_{(0)}\rangle} = \chi^{+}_{(0)} \quad (6.52)$$

$$Z_{\langle\tilde{\chi}_{((0,0),0)}||\chi^{+}_{(1)}\rangle} = \chi^{+}_{(1)}. \quad (6.53)$$

So we can discuss topological order referring to the characters with the implicit relation to the different boundary states present in the system. Also we point out that these BS should be associated to different kinds of linear defects compatible with conformal invariance.

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